Final Review 2

Problem 1: Consider the matrix:

$$A = \begin{bmatrix} -2 & 3 & 3 \\ 2 & -1 & -1 \\ -7 & 7 & 6 \end{bmatrix}$$

(a) Compute the characteristic polynomial of A.

(b) Compute eigenvalues and eigenvectors of A.

(c) Diagonalize the matrix A.

(d) Describe the behavior of the solution to the system $\dot{\boldsymbol{v}}(t) = A\boldsymbol{v}(t)$ as $t \to \infty$, where $\boldsymbol{v}$ is a vector consisting of three functions of t.

(e) Use Cramer's rule to find a solution to the equation $A\boldsymbol{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Problem 2: Consider the matrix:

$$A = \begin{bmatrix} \sqrt{2} & 0 \\ 1 & 1 \\ 0 & -\sqrt{2} \end{bmatrix}$$

(a) Compute the SVD of A.

(b) Consider the symmetric matrices $A^T A$ and AA^T. Are they positive definite, positive semi-definite, or neither? What are the corresponding energy functions of these symmetric matrices?

(c) Find numbers a and b for which the quantity:

$$(a\sqrt{2} - 1)^2 + (a + b - 1)^2 + (-b\sqrt{2} - 1)^2$$

is minimal.

Problem 3: Consider two random variables X and Y, which take values:

$$\{x = 1 \text{ and } y = 1\} \quad \text{with probability } \frac{2}{3}$$
$$\{x = 0 \text{ and } y = 3\} \quad \text{with probability } \frac{1}{3}$$

(a) Find linear combinations of x and y which are uncorrelated (i.e. have covariance 0).

(b) Find a general formula, in terms of matrices and vectors, for the covariance of any two linear combinations of these random variables: $ax + by$ and $a'x + b'y$ (where a, b, a', b' are numbers).